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OPTIMAL CONTROL OF THE DEFORMATION PATH OF A METAL UNDER COMPLEX LOADING†

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The problem of the optimal control of the deformation path of a metal under complex loading is formulated and solved. The equations in the theory of elastoplastic processes due to Il'yushin are used as the defining plasticity relations. A solution of the optimal problem is obtained using dynamic programming. It is shown that, by controlling the deformation paths, a significant reduction in the work of plastic deformation is possible. © 1999 Elsevier Science Ltd. All rights reserved.

It has been established experimentally [1, 2] that, in the complex loading of a metal, the force of plastic deformation is reduced compared with simple stretching or compression. This can be explained using Il'yushin's theory of elastoplastic processes [3]. The effect of complex loading on the magnitude of the work of plastic deformation and the possibility of using this effect to develop new technologies in the plastic treatment of metals are shown below.

1. FORMULATION OF THE PROBLEM

Consider the deformation of a homogeneous metal sample which is simultaneously under conditions of tension and torsion. This process is conveniently represented by a deformation path in Il'yushin vector space $\ni_1 - \ni_3$. Deformation paths of a specified length s^- will be investigated in this space. It is assumed that there is active loading of the sample in accordance with some law and that the initial segment of the path with a length s_0 corresponds to elastic loading. Only the plastic deformation of the sample is investigated. The point O at the start of the plastic deformation is adopted as the origin (Fig. 1). Suppose a vector $\boldsymbol{\sigma}$ is positioned at point O at an angle ϑ_0 to the $\ni_1$ axis. Among the paths of length $\bar{s}$, it is required to find such a path that the work of plastic deformation is a minimum, that is

$$A = \int\limits_{s_0}^{s_0+\bar{s}} \boldsymbol{\sigma} \cdot d\ni = \int\limits_{s_0}^{s_0+\bar{s}} \sigma(s)\cos\vartheta(s)ds \to \inf$$

The material is assumed to be strain-hardening, that is, $\sigma = \sigma(s)$, where σ is the modulus of the stress vector and ϑ is the angle between $\boldsymbol{\sigma}$ and $d\ni$ (the convergence angle).

We reduce this problem to a discrete form and approximate the deformation path with a multilink broken line, each link having the same length Δ. Then, $\bar{s} = N\Delta$, where N is the number of links.

As the control, we choose the angle of inclination of a link to the $\ni_1$ axis: $u(t) = \alpha(t)$ $(t = 1, \ldots, N)$ (Fig. 1). The phase variables $x_0(t) = s(t), x_1(t) = \ni_1(t), x_2(t) = \ni_3(t)$ $(t = 0, \ldots, N)$ are introduced. The scalar properties of the material (the modulus of the vector $\boldsymbol{\sigma}$ depend solely on the accumulated deformation s and, in the mth segment, they are independent of the previous controls $u(t)$ $(t = 1, \ldots, m - 1)$. The vector properties of the material (the angle ϑ) depend on the loading history. The complexity of the solution of the control problem therefore lies in the fact that it is necessary to take account of the deformation history.

According to the postulate of decaying memory [3], it is only necessary to remember the shape of the deformation path in a preceding segment of length λ (the trace of the lag of the material), the magnitude of which is determined experimentally for each metal. It is usually assumed that, in the case of steels, $\lambda = (6 \div 8)\varepsilon_s$, where $\varepsilon_s = \sigma_s/E$, σ_s is the yield stress of the material and E is the modulus of elasticity.

We consider the formulation of a control problem with hysteresis in a single step [4]. In this case, $\Delta = \lambda/2 = \text{const}$. The vector $\boldsymbol{\sigma}$ at the end of each link remains at a certain angle $\gamma(t)$ $(t = 2, \ldots, N)$ (Fig. 1) and we denote the path deflection angle by $\beta(t)$ $(t = 2, \ldots, N)$. It is then possible to determine

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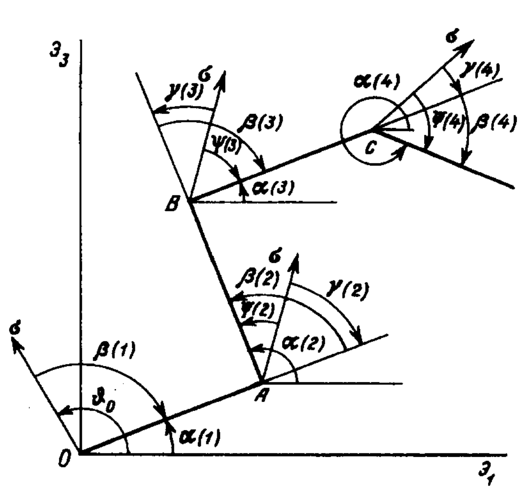

Fig. 1.

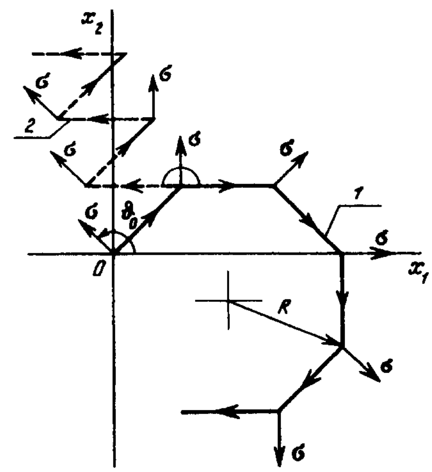

Fig. 2.

the angle of inclination of the vector $\boldsymbol{\sigma}$ at the beginning of each link, starting from the second, as follows:

$$\psi(t) = \gamma(t) + \beta(t), \quad t = 2, \ldots, N$$

The positive reading of each angle is shown in Fig. 1. The function of the angle of convergence ϑ at the tth step can be specified in the form [5]

$$\vartheta(y) = \psi(t)e^{-ky}, \quad y = s(t) - s(t-1) \tag{1}$$

where k is a constant of the material.

In this case, it is assumed in accordance with the lag principle, that $\vartheta(y) \to 0$ when $y \to \lambda$ and, in order to take account of the restricted memory, it is sufficient to introduce the two supplementary phase variables

$$x_3(t) = u(t), \quad t = 1, \ldots, N; \quad x_4(t) = x_3(t-1), \quad t = 2, \ldots, N$$

The deflection angle will then be determined in the following manner

$$\beta(t) = (u(t) - x_3(t-1)), \quad t = 1, \ldots, N$$

The angle $\psi(t)$ can be written in the form

$$\psi(t) = \gamma(t) + (u(t) - x_3(t-1)), \quad t = 1, \ldots, N$$
$$\gamma(t) = [\gamma(t-1) + (u(t-1) - x_3(t-2))]e^{-k\Delta}, \quad t = 2, \ldots, N$$

We will denote the convergence angle ϑ at the end of a link of length Δ by $x_5(t) = \gamma(t)$ ($t = 2, \ldots, N$).

The discrete optimal control problem can now be formulated as follows: it is required to find the optimal process $(\hat{\mathbf{u}}, \hat{\mathbf{x}})$ for which the functional

$$J(u,x) = \sum_{t=1}^{N} \int_{x_0(t-1)}^{x_0(t)} \sigma(y)\cos\{[u(t) - x_3(t-1) + (x_3(t-1) - x_4(t-1) + x_5(t-1))e^{-k\Delta}]e^{-ky}\}dy \tag{2}$$

attains a minimum value, that is

$$J(\hat{u}, \hat{x}) = \min_{u} J(u, x)$$

and constraints in the form of the equalities

$$x_0(t) = x_0(t-1) + \Delta, \quad x_1(t) = x_1(t-1) + \Delta\cos u(t), \quad t = 1,\dots,N$$
$$x_2(t) = x_2(t-1) + \Delta\sin u(t), \quad x_3(t) = u(t), \quad t = 1,\dots,N \tag{3}$$
$$x_4(t) = x_3(t-1), \quad x_5(t) = [x_5(t-1) + x_3(t-1) - x_4(t-1)]e^{-k\Delta}, \quad t = 2,\dots,N$$

are satisfied with boundary conditions

$$x_0(0) = s_0, \quad x_1(0) = 0, \quad x_2(0) = 0, \quad x_3(0) = \vartheta_0, \quad x_4(1) = \vartheta_0, \quad x_5(1) = 0 \tag{4}$$

and constraints in the form of the inequality

$$| u(t) - x_3(t-1) + [x_3(t-1) - x_4(t-1) + x_5(t-1)]e^{-k\Delta} | \leq \pi/2, \quad t = 1,\dots,N \tag{5}$$

The last condition follows from the requirement of active loading during the whole deformation process.
The discrete optimal control problem (2)–(5) satisfies Bellman's necessary condition for optimality [4], and we shall therefore use dynamic programming to solve it.

2. SOLUTION OF THE PROBLEM

We will use the following lemma.

Lemma. A minimum of the function

$$f(\beta) = \int_{s_1}^{s_2} \sigma(s)\cos[\beta e^{-k(s-s_1)}]ds$$

when $s_2 > s_1 \geqslant 0$, $k > 0$, $\sigma(s) > 0$ in the interval $\beta \in [-\pi/2, \pi/2]$ is reached when $\beta = \pm\pi/2$.
The proof of this lemma is rather obvious and follows from the properties of the function $\cos\beta$.
Using the well-known recurrence formula of dynamic programming [4] and the lemma, the solution of problem (2)–(5) can be written in the form

$$\hat{u}(N-p+1) = \pm\frac{\pi}{2} + \hat{x}_3(N-p)(1-e^{-k\Delta}) + (\hat{x}_4(N-p) - \hat{x}_5(N-p))e^{-k\Delta}, \quad p = 1,\dots,N \tag{6}$$

Using relation (3) together with boundary conditions (4), we obtain

$$\hat{u}(t) = \vartheta_0 \pm \frac{\pi}{2}(t-(t-1)e^{-k\Delta}), \quad t = 1,\dots,N \tag{7}$$

In this case, the deflection angle

$$\hat{\beta}(t) = \hat{u}(t) - \hat{u}(t-1) = \pm\frac{\pi}{2}(1-e^{-k\Delta}), \quad t = 2,\dots,N \tag{8}$$

In formulae (7) and (8), the sign in front of $\pi/2$ corresponds to the choice of sign in formula (6). It is then obvious that the optimal path is a broken line inscribed in a circle of a definite radius R (curve 1 in Fig. 2. The direction of motion along the path corresponding to a plus sign in front of $\pi/2$ in formulae (6) and (7) is shown by the arrows).
We shall now alternate the signs in front of $\pi/2$ in formula (6) and obtain the following optimal control function

$$\hat{u}(t) = \vartheta_0 \pm (-1)^{t-1}\frac{\pi}{2}e^{-k\Delta}, \quad t = 1,\dots,N \tag{9}$$

and deflection angle

$$\hat{\beta}(t) = (-1)^{t-1}\frac{\pi}{2}(1+e^{-k\Delta}), \quad t = 2,\dots,N \tag{10}$$

In this case the optimal path will be a zig-zag broken line (curve 2 in Fig. 2).
Similar calculations can be carried out using a smaller link length $\Delta = \lambda/3, \lambda/4, \dots$ and introducing additional phase variables. In this case, the form of solutions (7)–(10) does not change. However, in

the phase plane $x_1 - x_2$, the shape of the optimal path does undergo changes. For example, if one considers solution (7), (8), it can be seen that, as the link length Δ is reduced, the circumradius of the broken line with links of equal length also decreases and, in this case, the value of the functional (2) decreases.

We shall find the circumradius $\hat{R}$ which corresponds to the optimal path when $\Delta \to 0$. It is well known that the relation

$$\varphi_n = R_n^{-1}\lambda / n$$

holds for small central angles φ_n, where R_n is the circumradius and n is the number of links in the dashed line. In this case, the angle φ_n is equal to the path deflection angle $\beta(n)$. Then

$$\hat{R} = \lim_{n\to\infty} R_n = \frac{2\lambda}{\pi n(1 - e^{-k\lambda/n})} = \frac{2}{\pi k}$$

It is clear that the minimum of the functional (2) is equal to zero and that this value is reached in a circle of radius $\hat{R}$. However, this conclusion contradicts Il'yushin's plasticity postulate [6].

The contradiction obtained is obviously due to the fact that, in the case of a small link length ($\Delta \to 0$), formula (1) gives a significant error in the determination of the angle ϑ. It is therefore necessary to treat the choice of the approximating functions for the angle ϑ with care and to know the limits of their applicability. This, for example, relates to the well-known formula due to Dao-Zui-Bik [7] which describes the change in the convergence angle ϑ in a trajectory of constant curvature χ_0.

$$\vartheta(s_0, s) = \frac{s_0}{s}\left[\vartheta_0 - \frac{\chi_0}{k}\left(1 - \frac{1}{ks_0}\right)\right]e^{-k(s-s_0)} + \frac{\chi_0}{k}\left(1 - \frac{1}{ks}\right), \quad s \in [s_0, \bar{s}]$$

When $\vartheta_0 = \pi/2$, $\chi_0 = 1/\hat{R} = \pi k/2$, we obtain

$$\vartheta(s_0, s) = \frac{\pi}{2}\left[1 - \frac{1}{ks}(1 - e^{-k(s-s_0)})\right] = \frac{\pi}{2} g(s), \quad s \in [s_0, \bar{s}]$$

It can be shown that the values of the function $g(s)$ in the interval $[s_0, \bar{s}]$ when $\bar{s} \gg s_0$ are close to unity. For example, when $k = 300$, $s_0 = \varepsilon_s = 10^{-3}$, the form of the function $g(s)$ shown in Fig. 3 confirms that the value of the functional (2), when $\bar{s} \gg s_0$ is close to zero in deformation paths in the form of a circle of radius $\hat{R} = \pi k/2$.

Hence, the solution of the deformation path optimal control problem has shown that Il'yushin's theory of elastoplastic processes describes the effect of the reduction in the work of plastic deformation of a metal under complex loading. This result can be used to optimize practical technological processes involving the pressure treatment of metals. Although, in practical processes, it is usually required that a specified point in deformation space is reached, which results from the need to solve an optimal control problem with a clamped right-hand end, the qualitative form of the solution is retained in this case. For example, the deformation paths in the manufacture of a hollow steel cylinder by the backward extrusion method [8] have been investigated and it has been shown that, when using a scheme with "active friction", the deformation paths of the metal particles are distorted (they become zig-zag shaped) and a significant reduction in the extrusion force is observed.

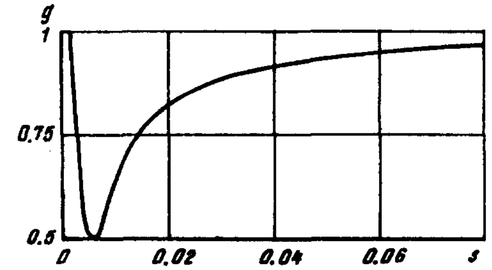

Fig. 3.

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